# Introduction to p-adic Langlands.

August 7, 2026

**§1** $f$ newform, wt. $k \geq 2$

$\rightsquigarrow \quad \pi = \pi_f$ auto. rep. : $\pi = \otimes'_v \pi_v$

$\pi_f \circlearrowleft GL_2(\mathbb{A}_\mathbb{Q})$, $\quad \pi_v \circlearrowleft GL_2(\mathbb{Q}_v)$

Associated Galois rep.

$$r_\pi : G_\mathbb{Q} \longrightarrow GL_2(\overline{\mathbb{Q}_p}) \text{ cts.}$$

Loc-glob. compat:

$$\forall \ell \neq p, \qquad \underbrace{r_\pi|_{D_\ell}}_{\text{WD rep.}} \overset{\text{LLC}}{\longleftrightarrow} \pi_\ell$$

Q: what about $\ell = p$ ??

$r_\pi|_{D_p} : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}_p})$ is de Rham

$\Leftrightarrow$ pot. semistable

$$D_{dR}(r_\pi|_{D_p}) \cong \text{"}D_{pst}\text{"}(r_\pi|_{D_p}) \qquad \text{2-dim}/\overline{\mathbb{Q}_p}$$

Hodge filtration jumps $0, k-1$ ??

Weil–Deligne reps.

$\updownarrow$ LLC

$\pi_p$.

Breuil's idea: want to build a $G \coloneqq GL_2(\mathbb{Q}_p)$ that "contains" the Hodge filt$^n$.

Take $\pi_p \otimes \mathrm{Sym}^{k-2}(\mathrm{std})$ not enough !

smooth $\circlearrowleft$ alg.

$G$

"locally alg".

$$\left\{ \begin{array}{c} G\text{-invariant norm on } \pi_p \otimes \mathrm{Sym}^{k-2} \\ \text{(up to equiv.)} \end{array} \right\} \overset{??}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Hodge filtr}^n \\ \text{compat. with given} \\ \text{WD rep., jumps} \\ 0, k-1 \end{array} \right\}$$

$$\Big\downarrow \text{complete}$$

$$\{ \text{unitary Banach space rep. of } G \}.$$

<u>Rk:</u> e.g. if $r_\pi$ is crystabelian, then $\exists!$ Hodge filtr$^n$ and $\exists!$ $G$-inv't. norm (Berger-Breuil)

<u>Rk:</u> cf. Breuil-Schneider conj.

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§2 <u>p-adic LLC for $GL_2(\mathbb{Q}_p)$</u>$=G$. $E/\mathbb{Q}_p$ finite ext. (large)

(Colmez, ...)

$$\{ G_{\mathbb{Q}_p} \xrightarrow{\text{cts.}} GL_2(E) \} \longrightarrow \left\{ \begin{array}{c} \text{(adm.)} \\ \text{unitary Banach reps.} \\ \text{of } G \text{ over } E \end{array} \right\}$$

$$\rho \longmapsto \Pi(\rho).$$

Fontaine $\quad \wr\!\wr \qquad \rightleftarrows$ Colmez functors

$$\{ (\varphi, \Gamma)\text{-modules} \}.$$

(1) $\rho \longmapsto \Pi(\rho)$ induces <u>bijection</u>

$$\left\{ \begin{array}{c} \text{abs. irred.} \\ \rho \end{array} \right\} / \cong \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{"supersingular" } \Pi \\ \text{irred.} \end{array} \right\} / \cong$$

(Paškūnas)

(2) If $\rho \sim \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$, $\chi_1 \chi_2^{-1} \neq \{ \underset{\text{cyclo.}}{\varepsilon^{\pm 1}} \}$, then $\quad B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$

$$0 \to \mathrm{Ind}_B^G (\chi_2 \otimes \chi_1 \varepsilon^{-1}) \to \Pi(\rho) \to \mathrm{Ind}_B^G (\chi_1 \otimes \chi_2 \varepsilon^{-1}) \to 0$$

$\chi_i : \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$ via LCFT.

(3) $\exists$ mod $p$ version: $\mathbb{F}/\mathbb{F}_p$ large

$$\{G_{\mathbb{Q}_p} \to GL_2(\mathbb{F})\} \longrightarrow \{\text{smooth rep. of } G \text{ over } \mathbb{F}\}$$
$$\bar{\rho} \longmapsto \Pi(\bar{\rho}).$$

(Breuil, "by hand")

(4) $\rho \longmapsto \Pi(\rho)$ is compatible with (semisimplified) mod $p$ reduction (Berger)

(5) $\rho$ is de Rham with distinct HT weights

$\Leftrightarrow$ $\Pi(\rho)$ contains a nonzero loc. alg. subrep.

(Breuil's idea ✓)

(6) <u>Local-global compat.</u>

completed coho. of modular curves : $\hat{H}^1 := \left( \varinjlim_{r \geq 0} H^1_{et}(X(Np^r)_{\bar{\mathbb{Q}}}, \mathcal{O}_E) \right)^{\wedge} \otimes_{\mathcal{O}_E} E$

(fixed on $N$; $p$-adic on the completion)

$G \times G_{\mathbb{Q}}$ acting on $\hat{H}^1$

$G$ (unitary Banach)

<u>Thm</u> (Emerton): If $r : G_{\mathbb{Q}} \to GL_2(E)$ abs. irred. s.t. + odd

$\bar{r}|_{D_p} \not\simeq \begin{pmatrix} 1 & * \\ & \bar{\varepsilon} \end{pmatrix} \otimes \chi \quad \forall \chi$, then

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(r, \hat{H}^1) \cong \Pi(r|_{D_p}) \underbrace{(\otimes \cdots)}_{\substack{\text{f.d.} \\ \text{(depends on } N)}}.$$

with $G$ acting on $\mathrm{Hom}_{G_{\mathbb{Q}}}(r, \hat{H}^1)$.

<u>Applications of $p$-adic LLC</u> (Kisin, Emerton):

- Fontaine-Mazur conj. for $GL_2/\mathbb{Q}$
- Breuil-Mézard conj.

§3 Hope: $p$-adic LLC generalises to e.g. $GL_n(K)$, $n \geq 1$, $K/\mathbb{Q}_p$ fin.

Challenges for $G := GL_2(K)$: even $K/\mathbb{Q}_p$ unram. deg. $f > 1$

1. Classification of irred. (adm.) smooth reps. of $G$ over $\mathbb{F}$ appears impossible (Breuil–Paškūnas)
2. The irred. supersing. reps. of $G$ over $\mathbb{F}$ are not of finite presentation (Schraen, Z. Wu)

Hope: Emerton's thm. generalises.

e.g. can define analogr of $\hat{H}^1$ using Shimura curves / tot. real fields $F$ s.t. $F_p \cong K$

and let $\Pi(r) := \mathrm{Hom}_{G_F}(r, \hat{H}^1)$.

Q (Locality): Does $\Pi(r)$ depend only on $r|_{D_p}$ ??
(up to f.d. factor)

Limited evidence:

Thm (Weight part of Serre's conj.; GLS, GK): Assume $\bar{r}$ modular.
The set of irred. $GL_2(\mathcal{O}_K)$-subreps. of $\Pi(\bar{r})$ only depends on $\bar{r}|_{D_p}$.

Thm (BHHMS): If $\bar{r}$ modular + $\bar{r}|_{D_p}$ "generic", then
$\Pi(\bar{r})$ is of finite length ($\leq f+1$) as $G$-reps.

# p-adic Langlands in families.

Taylor-Wiles-Kisin method.

Fix $\bar{r} : G_F \to GL_2(\mathbb{F})$ abs. irred. + modular.

Let $R_\infty := \underbrace{R_p}_{\text{univ. local defo. ring of } \bar{r}|_{D_p}} [\![ \underbrace{x_1, \dots, x_g}_{\text{auxiliary variables}} ]\!]$ with max. ideal $m_\infty$

CEGGPS: big patched module $\mathbb{M}_\infty$ is a f.g. $R_\infty[\![GL_2(K)]\!]$-mod.

s.t. $(\mathbb{M}_\infty / m_\infty)^\vee \cong \Pi(\bar{r})$ as $G$-reps.

Q (CEGGPS): Is $\mathbb{M}_\infty$ purely local, ie

$$\mathbb{M}_\infty \overset{?}{\cong} \underbrace{\mathbb{M}}_{\text{only depends on } \bar{r}|_{D_p}.} \otimes_{R_p} R_p[\![x_1, \dots x_g]\!].$$

($\Rightarrow$ prev. question)

Categorical p-adic LLC (Emerton-Gee-Hellmann)

($G = GL_n(K)$)

Hope "$\mathbb{M}$ spreads out over EG stack".

roughly, conj: $\exists$ functor

$$D^b(\underbrace{\text{sm.} G}_{\text{smooth reps.}}) \longrightarrow D^b_{coh}(\mathfrak{X}_n)$$