Introductory talk on cohomology of Shimura varieties

§ Modular curve.

<u>Recall</u>: ~~Y_1~~ $Y_1(N)$ modular curve.

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} \det = 1 \\ a,b,c,d \in \mathbb{Z} \end{matrix} ,\ \begin{matrix} a,d \equiv 1 \\ c \equiv 0 \end{matrix} \bmod N \right\}.$$

$$\coprod_{(\mathbb{Z}/N\mathbb{Z})^\times} \mathfrak{h}/\Gamma_1(N)$$

this is the key example to keep in mind.

We have an isomorphism

$$Y_1(N) \xrightarrow{\sim} GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) / K(N)$$

where $K(N) \subset GL_2(\mathbb{A}_f)$ compact open

$$= \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p \mid N} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}_p \ \begin{matrix} a,d \equiv 1 \\ c \equiv 0 \end{matrix} \bmod p^s,\ p^s \| N \right\}.$$

First observe: $Y_1(N)$ should really be $Y_1(N)(\mathbb{C})$

i.e. $\mathbb{C}$-points of an algebraic variety $/\mathbb{C}$.

this has a quasi-projective algebraic structure, using a canonical compactification $X_1(N)$.

$Y_1(N)$ has a model over $\mathbb{Q}$. even model integrally over $\mathbb{Z}[\frac{1}{N}]$.

this comes from the moduli interpretation of $Y_1(N)(\mathbb{C})$

$$= \left\{ \begin{matrix} \text{elliptic curves } E/\mathbb{C} \\ \text{together with point } P \in E \text{ of order } N \end{matrix} \right\} / \sim .$$

this moduli description makes sense for any $\mathbb{Z}[\frac{1}{N}]$-algebra, gives smooth scheme $/\mathbb{Z}[\frac{1}{N}]$. $N > 3$.

Definition of Shimura variety:

<u>Definition</u>: Shimura datum (G, X) $\quad G$ conn. reductive gp $/\mathbb{Q}$.

X $\quad G_{\mathbb{R}}$ conj-class of homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$.

such that:

(1) $\forall h \in X$, weights on $\mathrm{Lie}(G_{\mathbb{R}})$ given by $\mathrm{Ad} \circ h$ is

$$\{(-1,1), (0,0), (1,-1)\}.$$

(2) $\forall h \in X$, $\mathrm{ad}(h(i))$ defines a Cartan involution of $G^{ad}_{\mathbb{R}}$

(3) G^{ad} has no $\mathbb{Q}$-factor on which h is trivial.

idea: we want to form

$$\underline{G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K} \qquad K \subset G(\mathbb{A}_f) \text{ open compact subgroup.}$$

we want X to have structure of hermitian symmetric domain.

$\rightarrow$ this essentially what (1), (2) is asking for.

$X \simeq G(\mathbb{R})/A_\infty K_\infty \leftarrow$

maximal compact in $G(\mathbb{R})$

Rmk: If (G, X) not satisfying this conditions, eg. GL_3. then we can still contemplate the double quotient, call this locally symmetric space.

eg. Bianchi 3-manifold.

Rmk: For modular curve, $G = GL_2$

$X = GL_{2,\mathbb{R}}$-conj class of $h: a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

Shimura variety of level K

$$Sh_K(G,X)(\mathbb{C}) := \underline{G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K}.$$

this object $Sh_K(G,X)$ is a complex algebraic variety

Thm: (Deligne, Borovoi, Milne, ...)

$\exists$ canonical number field (depends only on (G,X), but not K)

E (call this reflex field) for which $Sh_K(G,X)$ has a model / E.

Consider compactly supported étale cohomology of $Sh_K(G,X)_E$

Rmk: can consider intersection étale cohomology.

completed cohomology
coherent cohomology
of Shimura variety.

$$\boxed{H^i_c(Sh_K(G,X)_{\bar{E}}, \bar{\mathbb{Q}}_\ell) \quad \ell \neq p. \quad \circlearrowleft \mathrm{Gal}(\bar{E}/E) \quad \circlearrowleft \Pi_f \leftarrow \text{adelic Hecke algebra.}}$$

Π_f : compactly supported $\bar{\mathbb{Q}}_\ell$-valued functions on $K\backslash G(\mathbb{A}_f)/K$. generated by indicator functions $g \in G(\mathbb{A}_f)$. 1_{KgK}

the action of $\underline{1_{KgK}}$ is by Hecke correspondences.

$$\begin{tikzcd} Sh_{K \cap gKg^{-1}}(G,X) \arrow[r, "\cdot g"] \arrow[d] & Sh_{g^{-1}Kg \cap K}(G,X) \arrow[d] \\ Sh_K(G,X) & Sh_K(G,X) \end{tikzcd}$$

Central question:

How to understand $H^i_c(Sh_K(G,X)_{\bar{E}}, \bar{\mathbb{Q}}_\ell)$ as a $\mathrm{Gal}(\bar{E}/E) \times \Pi_f$ module.

various conjectures (Arthur, Kottwitz) on how this looks like in terms of Langlands correspondence.

2 ways: 1) try to use $H^i_c(Sh_K(G,X)_{\bar{E}}, \bar{\mathbb{Q}}_\ell)$ to build a Langlands correspondence.

2) try to use a Langlands correspondence to understand $H^i_c(Sh_K(G,X)_{\bar{E}}, \bar{\mathbb{Q}}_\ell)$.

1) Langlands – Kottwitz method.
(point counting).

Basic observation: (Grothendieck–Lefschetz):

X smooth scheme / $\mathbb{F}_q$ dim d $q = p^n$, $\Phi_q = q$-frobenius.

$$\sum_{i=1}^{2d} (-1)^i \, tr(\Phi_q \mid H^i_c(X, \bar{\mathbb{Q}}_\ell)) = \# |X(\mathbb{F}_q)|.$$

to get a smooth scheme in char p, we have to consider the theory of integral models of Shimura varieties.

Setting: $K = K_p K^p$. K_p = hyperspecial subgp of $G(\mathbb{Q}_p)$. $G = G_{\mathbb{Q}_p}$.
$\qquad = G(\mathbb{Z}_p)$

Thm: (Kisin, Bakker–Shankar–Tsimerman, Madapusi–Youcis)

$\exists$ canonical integral model for $Sh_K(G,X)/_E$, $\mathcal{S}_K(G,X)$ over $\mathcal{O}_{E_v}$.
residue field k.

$v | p$ prime of E

when (1) (G,X) pre-abelian type

(2) $p >> 0$ otherwise.

$f \in \Pi_f$.

So we want to determine $\sum_{i=1}^{2d} (-1)^i tr(\Phi_q \times f^p \mid H^i_c(\mathcal{S}_K(G,X)_{\bar{k}}, \bar{\mathbb{Q}}_\ell))$.

$\qquad = (*)$.

$f^p \in$ ~~the~~ Π^p_f prime-to-p Hecke algebra. $\qquad$ # fixed points of this action of $\Phi_q \times f^p$ on $\mathcal{S}_K(G,X)_{\bar{k}}$.

${}_{K^p}\backslash G(\mathbb{A}^p_f)/K^p$.

Thm: (Kisin–Shin–Zhu): If (G,X) abelian type, then they verify this.
RHS expressed in terms of certain orbital integrals.

π automorph rep. $H^i_c(\mathcal{S}_K(G,X)_{\bar{k}}, \bar{\mathbb{Q}}_\ell)[\pi] \circlearrowleft \mathrm{Gal}(\bar{k}/k)$.

2. Applying categorical local Langlands.

Rough idea: usual local Langlands $\ell \neq p$.

F fin. extn of $\mathbb{Q}_p$.

$$\left\{ \begin{array}{c} \text{smooth ind.} \\ \overline{\mathbb{Q}}_\ell\text{-reps of} \\ G(F) \end{array} \right\} /\sim \quad \longleftrightarrow \quad \left\{ \begin{array}{c} L\text{-parameters.} \\ W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell) \end{array} \right\} / \hat{G}\text{-conj}$$

categorical local Langlands: upgrades this to map of derived categories.

instead of studying $D(G(F), \overline{\mathbb{Q}}_\ell)$

study $D(\underline{\mathrm{Bun}}_G, \Lambda)$ $\quad \Lambda$ coeffient. (Fargues-Scholze).

$D(\underline{\mathrm{Isoc}}_G, \Lambda)$ (Zhu).

geometric objects.

Bun_G = moduli space of G-bundle on Fargues-Fontaine curve.

Isoc_G = moduli space of isocrystals.

$$D(G(F), \Lambda) \subset D(\mathrm{Bun}_G, \Lambda)$$

or

$$D(\mathrm{Isoc}_G, \Lambda).$$

Key observation: Shimura variety is a fiber product (in a suitable setting) over Bun_G / Isoc_G.

separating info at p and away from p.

Thm: Have a 2-Cartesian diagram.

(Zhang, et al)

$$\begin{array}{ccc} Sh_K(G,X)^{\circ,\diamond} & \hookrightarrow & \underline{Sht(G,\mu)} \\ \downarrow & \ulcorner & \downarrow \\ \underline{Igs_{K^p}(G,X)} & \xrightarrow{\pi_{HT}} & \underline{\mathrm{Bun}_G} \end{array}$$

away from p contribution $\to$ Igusa stack.

Igs on $D(\mathrm{Bun}_G)$

$= \underline{\pi_{HT!}\Lambda}$.

Key consequence: $D(\mathrm{Bun}_G, \Lambda)$ has a spectral action

T_μ action on $D(\mathrm{Bun}_G, \Lambda)$ for all $\mu \in X_*^+(T)$.

we have an isomorphism

$\mu \rightsquigarrow$ obtained from X.

$$R\Gamma_c(Sh) = j_i^* \underline{T_\mu}(Igs)[\text{some shift}].$$