Moduli of p-adic Galois representations

$\rightarrow$ EG survey notes

$\rightarrow$ Pham rk 2 case

§ Intro

Goal: Study continuous $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) =: G_{\mathbb{Q}_p} \rightarrow GL_n(\mathbb{Z}_\ell)$

$\downarrow$

appear naturally $H^*_{\acute{e}t}(X_{\mathbb{Q}_p} \times \overline{\mathbb{Q}}_p, \mathbb{Z}_\ell)$ $\leftarrow$ relevant

or restriction of $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ of $X_{\mathbb{Q}}$, $\overline{\mathbb{Q}}$

- for modularity lifting
- local p-adic Langlands
- FM conjecture (Kisin, Emerton, Pan)

Hope: $\exists$ "geometric space" $\mathcal{X}_{\mathbb{Q}_p, GL_n/\mathbb{Z}_\ell}$

s.t. $\mathcal{X}(\mathbb{Z}_\ell) = \{\rho \text{ continuous}\}/\text{isomorphism}$

Key difficulty: continuity condition

- Characters: $G^{ab}_{\mathbb{Q}_p} = \mathbb{Z}_p^\times \times p^{\hat{\mathbb{Z}}} = \varprojlim$

$\cup$

$WD^{ab}_{\mathbb{Q}_p} = \mathbb{Z}_p^\times \times p^{\mathbb{Z}} = \varprojlim_n \underbrace{(\mathbb{Z}/p^n)^\times}_{\text{finite group}} \times p^{\mathbb{Z}}$

hence $\varinjlim_n \mathrm{Spec}\left(\mathbb{Z}_\ell[X]/X^{p^n}-1\right) \times \mu_{p-1} \times \mathbb{G}_{m,\mathbb{Z}_\ell}$ f.t. scheme representing $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{G}_m)$

étale$/\mathbb{Z}_\ell$ if $\ell \neq p$ ($X^{p^n}-1$ is separable)

nilpotent thickening if $\ell = p$

- Algebraize unramified characters

$\mathrm{Hom}^{c^0}\left(G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}, \overline{\mathbb{F}}_p^\times\right)$, $G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p} \cong \hat{\mathbb{Z}} \ni p$: naive: algebraic family$/\overline{\mathbb{F}}_p$

i.e. $= \mathrm{Spec}(A)(\overline{\mathbb{F}}_p)$

$A = \overline{\mathbb{F}}_p[X, X^{-1}]$

This gives a "universal family" $\mathrm{unr}_X : p^{\mathbb{Z}} \rightarrow A^\times$, $p \mapsto X$ but is not c^0!

- If such algebraic family exists, what are its specializations?

e.g. $X/\mathrm{Spec}\,\mathbb{F}_p$ s.t. $X(\overline{\mathbb{F}}_p) \simeq \{0 \rightarrow \mathbb{1} \rightarrow * \rightarrow \mathrm{unr}_\lambda \omega \rightarrow 0\}/\text{iso}$ (c^0; ω = mod p cyclo char)

rk 2 vector bundle V_p

$\rightsquigarrow$ there $\exists$ a universal $G_{\mathbb{Q}_p} \rightarrow \begin{pmatrix} \mathbb{1} & * \\ 0 & \mathrm{unr}_\lambda \omega \end{pmatrix}$ over $\mathbb{F}_p$ (recall $H^1(G_{\mathbb{Q}_p}, \omega)$ is 1-dim'l)

$\rightsquigarrow \exists$ a c^0 $0 \rightarrow \mathbb{1} \rightarrow V_p/\mathbb{F}_p(T) \rightarrow (\mathrm{unr}_\lambda \omega)/\mathbb{F}_p(T) \rightarrow 1$ $\boxed{\text{indecomposable}}$

hence $\mathrm{Spec}(\mathbb{F}(T)) \rightarrow \underbrace{Gr(1, V_p)}_{\text{proper}/X}$

→ extends to Spec($\mathbb{F}[[T]]$) i.e. 1 distinguished $G_{\mathbb{Q}_p}$-invariant line (out of 2 as $T=0 \leftrightarrow 1 \oplus \mathrm{unr}_\lambda \omega$)

Another problem:

$0 \to \mathbb{1} \to \mathcal{V}_p \to \mathrm{unr}_\lambda \omega_{/\mathbb{F}_p[[T]]} \to 0$ with $\mathcal{C}^0$-action should have a $G_{\mathbb{Q}_p}$-invariant lattice

counterexample: if $\rho_c = \begin{pmatrix} 1 & c(g) \\ & \mathrm{unr}_\lambda \omega \end{pmatrix}$, $c \in Z^1(G_{\mathbb{Q}_p}, \mathrm{unr}_\lambda \omega)$

then $\begin{pmatrix} 1 & T^{-1} \cdot c(g) \\ & \chi \cdot \mathbb{F}_p[[T]] \end{pmatrix} \in H^1(G_K, \bar{\chi}, \mathbb{F}(T))$ but does not lift to $\mathbb{F}_p[[T]]$

Intuitive fix: $H^1_c(G_{\mathbb{Q}_p}, \bar{\chi}) \circlearrowleft \mathbb{G}_m$-action by $c \mapsto t^2 c$

1-dim^{al} / $\mathbb{F}_p$

here $X(\mathbb{F}_p) \cong \mathbb{F}_p / \mathbb{F}_p^\times = \{0, *\}$
$\downarrow$
generic pt, non-closed

§ The Emerton–Gee approach

Thm (Fontaine): Let $R \in \mathrm{CLN}_{\mathbb{Z}_p}$. Then

$$\{\mathcal{C}^0\ \rho : G_{\mathbb{Q}_p} \to GL_n(R)\} \underset{\sim}{\rightleftarrows} \left\{\begin{array}{l}\text{rk } n \text{ projective,} \\ \text{étale } (\varphi, \Gamma)\text{-modules} \\ \text{over } R\end{array}\right\}$$

$\mathbb{A} := \left(\mathbb{Z}_p[[X]]\left[\frac{1}{X}\right]\right)^{\wedge_p}$ ← p, X-adic topology

$\hat{\mathbb{A}}^{\mathrm{ur}} :=$ integers of $\left((\mathbb{A}[\tfrac{1}{p}])^{\mathrm{unr}}\right)^\wedge$

M: $(\varphi, \Gamma \cong \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \xrightarrow[\sim]{\text{cyclotomic character}} \mathbb{Z}_p^\times)$-module.

over $\mathbb{A} \hat{\otimes}_{\mathbb{Z}_p} R$ + $\mathcal{C}^0$ semilinear $\varphi_M : M \to M$ w.r.t. φ $X \mapsto (1+T)^p - 1$

$\mathcal{C}^0$ semilinear $\gamma_M : M \to M$ w.r.t. $X \mapsto (1+X)^{\chi_{cy}(g)} - 1$

s.t. $\varphi^* M \to M$ is iso & $\varphi_M \circ \gamma_M = \gamma_M \circ \varphi_M$

(equivalence: $\rho \mapsto (\rho \otimes \hat{\mathbb{A}}^{\mathrm{ur}})^{\ker(G_{\mathbb{Q}_p} \to \Gamma)}$, $(M \otimes \hat{\mathbb{A}}^{\mathrm{ur}})^{\varphi = 1} \leftarrow M$)

<u>Rem</u>: 1) (Herr) $H^1\left(0 \to M \xrightarrow[\binom{\varphi-1}{\gamma-1}]{} M \oplus M \xrightarrow[((\varphi-1),\ \gamma-1)]{} M \to 0\right)$

$\simeq H^1(G_{\mathbb{Q}_p}, \rho_M)$

2) $H^1_c\left((\varphi,\Gamma),\ (A_{/p})\hat{\otimes}_{\mathbb{F}_p}\mathbb{F}[T]\right) \to$

$\to H^1_c\left((\varphi,\Gamma), A_{/p}\hat{\otimes}\mathbb{F}(T)\right)$

$\to H^1_c\left((\varphi,\Gamma), A_{/p}\hat{\otimes}\frac{\mathbb{F}(T)}{\mathbb{F}[T]}\right) = 0$

$\Rightarrow$ find lattices$/_{\mathbb{F}[T]}$ for the universal extension of (φ,Γ)-modules

i.e. if $\rho_c \leftrightarrow M_c$, $\mathrm{Mat}(\varphi_{M_c}) = \begin{pmatrix} 1 & \alpha \\ & \lambda\frac{1}{X^i} \end{pmatrix}$

$\mathrm{Mat}(\gamma_{M_c}) = \begin{pmatrix} 1 & \beta \\ & 1+ \end{pmatrix}$ $\times \mathbb{F}[[X]]$

then $\begin{pmatrix} 1 & \alpha/T \\ & \lambda/X^i \end{pmatrix}$ is the universal ext$^{\underline{e}}$

$\downarrow$

Change of basis $\begin{pmatrix} 1 & \\ & T \end{pmatrix}$ to have ~~[illegible]~~ the lattice

Exemple 3: (φ,Γ)-module$/_{\mathbb{F}_p[[T]]}$

$\mathrm{Mat}(\varphi_M) = \begin{pmatrix} T & -1 \\ \lambda\ i & 0 \end{pmatrix}$ $p-2 > i > 1$ (have T^1 action by looking at $\begin{pmatrix} 1 & * \\ & \chi \end{pmatrix}$ ext$^{\underline{n}}$)

over ~~[illegible]~~ $\mathbb{F}_p[[X]]((T))$: $\exists$ solution to $\underset{\mathbb{F}[[X]]}{Z} = T^{-1}(X^i + Z^{p+1})$

$\Rightarrow$ $e_1 \mapsto e_1 + Z e_2$, $e_2 \mapsto e_2$ change of basis $\begin{pmatrix} T-Z^p & -1 \\ & Z \end{pmatrix}$

Again: if $M \leftrightarrow$ universal $\rho_T: G_{\mathbb{Q}_p} \to GL_2(\bar{\mathbb{F}}_p)$

here $\mathrm{Spec}(\mathbb{F}(T)) \to \mathcal{G}r(1, \bar{\mathbb{F}}_p)$

by properness, $\rho_{T=0}$ should be reducible.

Emerton-Gee:

functor $\mathcal{X}_{\mathbb{Q}_p, GL_n}$: p-adically complete $\mathbb{Z}_p$-alg $\to$ Groupoid

$A \longmapsto (\varphi\text{-}\Gamma)\text{-Mod}^{ét}_{A, \text{rk } n}$

morphisms : isomorphisms

e.g. $\left[* / GL_n \right](A) := \left\{ \begin{matrix} \text{rk } n \\ \text{projective } A\text{-module} \end{matrix} \right\} / \text{isom.}$

+ sheaf axioms

Fontaine's $\Rightarrow$ $\mathcal{X}(\mathbb{F}_p) \simeq \{ \mathcal{C}^0 \ \rho : G_{\mathbb{Q}_p} \to GL_n(\mathbb{F}_p) \} / \text{iso.}$

Prop: Given $\bar{\rho} \in \mathcal{X}(\mathbb{F}_p)$, there exist $\mathrm{Spf}(R^{\square}_{\bar{\rho}}) \longrightarrow \mathcal{X}$ versal i.e.

$$\begin{tikzcd} \mathrm{Spf}(R^{\square}_{\bar{\rho}}) \arrow[r] & \mathcal{X} \\ \mathrm{Spec}(A_0/I) \arrow[u] \arrow[r] & \mathrm{Spec}(A/I) \arrow[u] \arrow[ul, dashed, "\exists"] \end{tikzcd}$$

artinian local

Pb: is $\mathcal{X}$ a "reasonable object"

Thm: $\mathcal{X}$ is an Ind-algebraic stack, Noetherian

Pf: 1. $\mathcal{X}$ is Ind-algebraic

2. $\mathcal{X}_{\mathbb{F}_p, \text{red}}$ is algebraic & finite presentation

$\Rightarrow \mathcal{X}$ is formal algebraic $\underset{\text{abstract nonsense}}{\Rightarrow}$ $\mathcal{X}$ is Noeth.

① "discretization of Γ"

$\mathcal{C}^{(R)}_h$ = Groupoid of rk n projective A^+_R-modules with $\varphi_M : M \to M$ φ-semilinear & Coker(φ_M) killed by F^h

$F \in \mathbb{Z}_p[X]$ deforming $X \mapsto X^p$ mod p

endomorphism on A^+

Pappas-Rapoport (φ-torsion), Emerton-Gee : $\mathcal{C}_h$ is p-adic formal algebraic

$\rightsquigarrow$ forming scheme theoretic images: $\mathcal{C}_h \twoheadrightarrow \mathcal{R}_h \subseteq \mathcal{R}$

invert X & complete

stack of rk n étale φ-modules $\hat{A}$

Thm EG: $\mathcal{R} = \varinjlim_h \mathcal{R}_h$ & $\mathcal{R}_h$ is algebraic $\forall h$.

(rough idea: find φ-stable lattices in étale φ-mod)

cf. later

Formal arguments: $\mathcal{R}^{disc}$:= rk n projective étale $(\varphi, T^{disc} = \langle\gamma\rangle)$-mod$/A$ is Ind-algebraic

$T^{disc} \cap T = \langle\gamma\rangle$

$\cong \mathcal{R} \times_{s, \mathcal{R}\times\mathcal{R}, x\mapsto(x,\gamma(x))} \mathcal{R}$

Lemma: $M \in \mathcal{R}^{disc}(R/p^a)$

then $T^{disc} \curvearrowright M$ extends to $T \curvearrowright_{\mathcal{E}^0} M$

$\Leftrightarrow$

$\exists$ A^+-lattice $\mathfrak{M} \subseteq M$ s.t. $(\gamma^{p^a}-1)(\mathfrak{M}) \subseteq \ldots$

$\exists s \quad \subseteq X \cdot \mathfrak{M}$

$\{$moduli of such lattices, coker$(\varphi_{\mathfrak{M}}) X^h = 0\}$ Closed condition

$\Rightarrow \mathcal{W}_{h,s} \xrightarrow{\ } \mathcal{R}^{disc} \times \mathcal{C}_h$ is a closed immersion

$(\otimes A, \text{forget } T^{disc})$

p-adic formal algebraic

$\mathcal{W}_{h,s}$ & $\varinjlim_s \mathcal{W}_{h,s}$ are

Final step:

$$\begin{tikzcd} \mathcal{W}_h \arrow[r] \arrow[d, "\otimes A"] & \mathcal{R}^{disc} \times \mathcal{C}_h \arrow[d] \\ \mathcal{X}_n \arrow[r, "\text{restriction to } T^{disc}"'] & \mathcal{R}^{disc} \end{tikzcd}$$

& extend the action

Prove that $\varinjlim_h$ ($\mathrm{Im}(\mathcal{W}_h \to \mathcal{R}^{disc})$).

by showing $\mathrm{Im}(\mathcal{W}_h \to \mathcal{R}^{disc}) \subseteq \mathcal{X}$

Lemma: If $M \in$ Image( ) then $\exists$ a φ-invariant A^+-lattice.

② Algebraicity of $\mathcal{X}_{\bar{\mathbb{F}}_p, red}$

Idea: Cover $\mathcal{X}_{\bar{\mathbb{F}}_p, red}$ by "family" of extensions of (φ, T)-modules

$n=2$ reducible $\bar{\rho} \sim \begin{pmatrix} 1 & * \\ & \underbrace{\mathrm{unr}_\lambda \cdot \omega^i}_{\chi_\lambda} \end{pmatrix}$ $i = 0, \ldots, p-2$, $\lambda \in \bar{\mathbb{F}}_p^\times$

rk 1 universal (φ, T) module up to twist

$\Rightarrow \mathrm{Ext}^1(\mathbb{1}, \chi_{\mathbb{G}_m})$ line bundle$/\mathbb{G}_m$ (if $i \not\equiv -1, 0$ mod $p-1$)

$\mathcal{Z} \simeq \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{G}_m$ by adding an unram. twist + $\mathbb{G}_m \times \mathbb{G}_m$-action (by change of basis)

$\underbrace{\left[(\mathbb{A}^1 \times \mathbb{G}_m \times \mathbb{G}_m) / \mathbb{G}_m \times \mathbb{G}_m\right]}_{\text{1-dimensional}} \xrightarrow{\text{monomorphism}} \mathcal{X}_{GL_2} =: \mathcal{X}_{i,j}$

More generally stratify $\mathbb{G}_m$ s.t. $\mathrm{Ext}^2(\mathbb{1}, \mathcal{X}_{\mathbb{G}_m})$ is a vector bundle

$\Rightarrow \underbrace{\mathcal{Z}^1}_{\substack{\text{vector bundle}/\mathbb{G}_m \\ \text{rk } 2}} \to \mathrm{Ext}^1(\mathbb{1}, \mathcal{X}_{\mathbb{G}_m})$ giving rise $\mathcal{Z}^1 \times \mathbb{G}_m \xrightarrow{\text{add unram. twist}} \mathcal{X}_2$

$\to$ kernel on fibers has dim 1

Again $\dim(\mathrm{Im}) = \underset{\substack{\| \\ \mathbb{G}_m \times \mathbb{G}_m}}{4-2} \underbrace{-1}_{\text{dim ker of fibers.}}$

Consequence: $\mathcal{X}_{GL_2, red} = \bigcup_{\substack{i=0,\dots,p-1 \\ j=0,\dots,p-2}} \mathcal{X}_{i,j} \cup \bigcup \{\text{0-dimensional}\}$ 1-dim^{el} $(\mathrm{ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^r)$ $\otimes$ unr

with bijection irr. $GL_2(\mathbb{F}_p)$-reps $(\mathrm{Sym}^i \mathbb{F}_p^2) \otimes \det^j$

$\Rightarrow \mathcal{X}_{GL_2, red}$ is algebraic (as $\bigcup$ of algebraic)

Applications

Crystalline substacks & BM conjecture

$A^+ = \mathbb{Z}_p[\![X]\!] \hookrightarrow W(\mathcal{O}_{\mathbb{C}_p}^\flat) = A_{\inf}$

$X \longmapsto [p^\flat]$ p-power roots of p

BK-module $M \longrightarrow M \otimes A_{\inf}$: BKF-module

$\downarrow$

$M \otimes W(\mathbb{C}_p^\flat)$ $\{$is an étale (φ, G_K)-module$\}$ $\overset{EG}{\updownarrow}$ $\{$étale (φ,Γ)-modules$\}$

Capture crystalline condition:

Given $\mathfrak{M} \subseteq \mathfrak{M}^{\inf}$ $\swarrow$ BKF G_K-mod, φ-stable A^+-lattice

s.t. $(g-1)(\mathfrak{M}) \subseteq \varphi^{-1}([\varepsilon]-1)[p^\flat]\mathfrak{M}^{\inf}$ $(*)$

$\Rightarrow$ define $\mathcal{C}_h^{crys} \xrightarrow{- \otimes W(\mathbb{C}_p^\flat)} \mathcal{X}$

height of BKF G_K-modules

deduce from $\mathcal{C}_h + (*)$ closed condition

$\mathcal{C}_h^{crys}$ is p-adic formal algebraic

Thm: $\mathrm{Im}(\mathcal{C}_h^{\mathrm{crys}} \to \mathcal{X})$ is p-adic formal algebraic f.t. & flat $/\mathbb{Z}_p$

its $\mathbb{Z}_p$-pts $\leftrightarrow$ $\rho: G_{\mathbb{Q}_p} \to GL_n(\mathbb{Z}_p)$ which are crystalline, HT weights $\in [0,h]$.

$\hookrightarrow$ equidim^{l} of dim $\mathcal{X}_{\overline{\mathbb{F}}_p}$

BM Conjecture:

$$Z(\mathcal{X}^{\mathrm{crys}, \lambda}_{\overline{\mathbb{F}}_p}) = \sum_\sigma [W(\lambda - \begin{pmatrix} n-1 \\ n-2 \\ \vdots \\ 0 \end{pmatrix}) : \sigma] \cdot Z_\sigma$$

alg. rep^n of highest weight $\lambda - \begin{pmatrix} n-1 \\ n-2 \\ \vdots \\ 0 \end{pmatrix}$

Serre weight

universal cycles, containing $\mathcal{X}_\sigma$

Existence of crystalline lifts

Thm: Let $0 \to \bar\rho_n \to \bar\rho_{n+1} \to \bar\chi \to 0$

having a crystalline lift ρ_n° HT λ

having a crystalline lift HT $\ll \lambda$

Then $\exists$ $0 \to \rho_n'^{\circ} \to \rho_{n+1}^\circ \to \chi \to 0$

cr. wt λ same crys. char

crystalline

Idea ($n=1$): Obstruction: (via $0 \to \mathbb{Z}_p(\chi) \to \mathbb{Z}_p(\chi) \to \mathbb{F}_p(\chi) \to 0$)

$$H^2(G_K, \chi)[p] \underset{\text{dual}}{\longleftrightarrow} H^0(G_K, \mathbb{Q}_p/\mathbb{Z}_p(\chi_{cyc}\chi^{-1}))/p$$

Problems when $\bar\chi = \omega$: $H^1(G_K, \omega)$: 2 dim^{al}

$H^1(G_K, \chi_{cyc}^p) \otimes \mathbb{F}_p$ 1-dim^{al}

$\Rightarrow$ Consider $\chi_t = \chi_{cyc}^p \cdot \mathrm{un}_{1+T} / \mathbb{Z}_p[[T]]$

Check $\forall x \in \mathrm{Hom}(\mathbb{Z}_p[[T]], \mathbb{Z}_p)$,

$$0 \to H^1(G_{\mathbb{Q}_p}, \chi_t) \otimes_{\mathbb{Z}_p[[t]], x} \mathbb{Z}_p \to H^1(G_{\mathbb{Q}_p}, \chi_t \otimes_x \mathbb{Z}_p) \to \mathbb{Z}_p[[t]]/(p, t)$$

i.e. $H^1(G_K, \chi_t) = \ker\left(\mathcal{O}[[t]]^{\oplus 2} \xrightarrow{(p,t)} \mathcal{O}[[t]]\right)$

non-principal

$\rightsquigarrow$ blow up of (p,t): $0 \to \ker \to * \to Bl(p,t) \to 0$

mult. ideal

invertible

$\Rightarrow$ splits locally Zariski $\rightsquigarrow$ get lift